Definitions in Topology

A **topological space** is a set X of **points** and a collection $\mathcal{U} \subseteq 2^X$ of **open sets** such that:

- $\emptyset, X \in \mathcal{U}$.
- If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
- For any set I and any function $f : I \to \mathcal{U}$, $\bigcup_{i \in I} f(i) \in \mathcal{U}$.

Such an admissible $\mathcal{U}$ is called a **topology**. A set is **closed** if it is the complement of an open set, and **clopen** if it is both closed and open. A set N is a **neighbourhood** of a point x if there is some open set U such that $U \subseteq N$ and $x \in U$. Alternatively, it is a neighbourhood of a set E if there is some open set U such that $E \subseteq U \subseteq N$. A point x is a **limit point** of a set E if every neighbourhood of x contains a point in $E \setminus x$.

A sequence (x_n) in X **converges** to a point x if for all open neighbourhoods U of x there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U$.

If X and Y are topological spaces, the function $f : X \to Y$ is **continuous** if for any open set $U \subseteq Y$, its inverse image $f^{-1}(U) \subseteq X$ is also open. f is a **homeomorphism** if it is invertible and if f^{-1} is also continuous.

A topology $\mathcal{U}$ is **generated** by $B \subseteq 2^X$ if it is the smallest topology containing B. We say that the generating set B is a **base** of the topology if also every point in X is contained in an element of the base, and for every $B_1, B_2 \in B$, for every $x \in B_1 \cap B_2$ there exists some $B_3 \in B$ such that $x \in B_3 \subseteq B_1 \cap B_2$. A **local base** for a point x is a collection of neighbourhoods of x such that any other neighbourhood of x contains an element of the base.

If X is a topological space and $Y \subseteq X$, the **subspace topology** on Y is $\{V \in Y \mid V = U \cap Y, U \subseteq X,\ U \text{ open}\}$. If X_i is a family of topological spaces for $i \in I$, then the **product space** $X = \prod_i X_i$ is the Cartesian product of the spaces X_i, with the topology generated by the sets $\prod_{i \in I} U_i$ where each U_i is open in X_i and $U_i \neq X_i$ for only finitely many i. If $\sim$ is an equivalence relation on X then the **quotient space** $X/\sim$ is the set $\{[x] = \{y \in X \mid y \sim x\} \mid x \in X\}$ with the topology $\{U \subseteq X/\sim \mid \bigcup_{[x] \in U} \bigcup_{y \in [x]} y \text{ open in } X\}$.

A space X is **Kolmogorov**, or $\mathbf{T_0}$, if for every pair of distinct points at least one has a neighbourhood not containing the other. It is **Fréchet**, or $\mathbf{T_1}$, if each of the two points has a neighbourhood not containing the other. It is **Hausdorff**, or $\mathbf{T_2}$, if the neighbourhoods can be made disjoint. It is **Urysohn**, or $\mathbf{T_{2.5}}$, if the disjoint neighbourhoods can additionally be made closed. It is **completely Hausdorff** if there exists a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$. It is **regular** if for any closed set C and any point $x \notin C$, x and C have disjoint neighbourhoods, and it is **regular Hausdorff** or $\mathbf{T_3}$ if it is regular and Hausdorff. It is **completely regular** if there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(Y) = \{1\}$, and **Tychonoff** or $\mathbf{T_{3.5}}$ if it is completely regular and Hausdorff. A space is **normal** if every pair of disjoint closed sets C and D have open neighbourhoods, it is **completely normal** if every subspace is normal, and it is **perfectly normal** if there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(\{0\}) = C$ and $f^{-1}(\{1\}) = D$. Spaces that are normal and Hausdorff are called $\mathbf{T_4}$, spaces that are completely normal and Hausdorff are called $\mathbf{T_5}$, and spaces that are perfectly normal and Hausdorff are called $\mathbf{T_6}$.

A subset E of X is **sequentially open** if for all $x \in E$ and all sequences (x_n) converging to x, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in E$. A space is **sequential** if every sequentially open subset is open. A subset E of X is **dense** if every open set in X has a non-empty intersection with E, and X is **separable** if it has a countable dense subset. A space is **first-countable** if every point has a countable local base, it is **second-countable** if the topology has a countable base.

A **cover** of a space is a collection of sets whose union is the whole space, and a **sub-cover** is a sub-collection of a cover which itself is also a cover. A **refinement** of a cover is a new cover whose elements are all subsets of elements of the original cover. A cover is **locally finite** if every point in the space has a neighbourhood that intersects only finitely many sets in the cover.

A space is **Lindelöf** if every cover of open sets (or open cover) has a countable sub-cover, and is **compact** if the sub-cover is finite. **Countably compact** only need have finite sub-covers for countable open covers. Spaces are **paracompact** if open covers have open refinements which are locally finite. A space is **σ-compact** if it has a countable cover by compact subspaces, and **locally compact** if every point has a compact neighbourhood. It is **compactly generated** if subset A is closed iff for all compact subspaces K, $A \cap K$ is closed in K. A **sequentially compact** space is one where every sequence has a subsequence that converges. A **pseudocompact** space is one such that its image under any continuous function mapping it to $\mathbb{R}$ is bounded, and a **limit point compact** space is one where every infinite set has a limit point.

A space is **metrisable** if it is homeomorphic to a metric space, and **locally metrisable** if every point has a metrisable neighbourhood.

A **disconnected** space is one which has a non-trivial clopen subset, and a **connected** space is one which is not disconnected. A space X is **totally disconnected** if it has no non-trivial connected subsets, and **totally separated** if for any points x and y, there are disjoint open neighbourhoods U of x and V of y such that $X = U \cup V$. X is **path-connected** if for any two points x and y, there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$, and **arc-connected** if f is a homeomorphism between $[0, 1]$ and $f([0, 1])$. A space is **simply connected** if every pair of paths between two points can be continuously transformed into each other, more formally, if for any continuous map f from the unit circle in $\mathbb{R}^2$ to X, there exists a continuous map F from the unit disc in $\mathbb{R}^2$ to X such that $F = f$ when restricted to the unit circle. A space is **hyperconnected** if no two non-empty open sets are disjoint. A **locally connected** space is one where every point has a local base of open connected sets, and a **locally path-connected** space is one where the local base is of open path-connected sets.