A nice representation of the Laplacian

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The Laplacian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the sum of its second derivatives along each dimension:

$$\Delta f(x) = \sum_i \frac{\partial^2 f(x)}{\partial x_i^2}.$$

In this document, I will show a proof of the following theorem:

Theorem 1. *The Laplacian of a function at point x is the limit of the difference between the average value of the function on a sphere of radius r around x, and the value of the function at x itself, multiplied by $2n/r^2$ where n is the dimension of the space. That is,*

$$\Delta f(x) = \lim_{r \to 0} \frac{2n}{r^2} \left(\frac{\int_{S(x,r)} f(x')\,\mathrm{d}\sigma(x')}{\mathrm{Vol}_{n-1}(S(x,r))} - f(x) \right)$$

where $S(x,r)$ is the sphere centred at x at radius r, σ is the $(n-1)$-dimensional surface measure on the sphere, and Vol_{n-1} is the $(n-1)$-dimensional volume of a set.

This theorem closely connects the Laplacian operator to the graph Laplacian discussed in e.g. [1], motivating the definition for the latter. I believe that I came across a proof of this on the internet at some point in the past, but have not been able to find this proof anywhere, and so endeavoured to provide it myself.

As motivation for the theorem, consider the case when $n = 1$. Functions with a positive second derivative are convex: the value at a point is less than the average of the values at 'neighbouring points', and a similar thing is true for functions with a negative second derivative. It also relates to the 'spherical mean property' of harmonic functions (that is, functions f that satisfy $\Delta f = 0$).

The following lemma deals with the gamma function, which will pop up in the proof of our main theorem only to eventually factor out.

Lemma 2.

$$2\int_0^\infty y^n \exp(-y^2)\,\mathrm{d}y = \Gamma\left(\frac{n+1}{2}\right).$$

Proof. Use variable substitution. Let $t = y^2$. Then, $\mathrm{d}t = 2y\,\mathrm{d}y$. The bounds of integration do not change.

$$\begin{aligned}
2\int_0^\infty y^n \exp(-y^2)\,\mathrm{d}y &= \int_0^\infty y^{n-1}\exp(-y^2)\times 2y\,\mathrm{d}y \\
&= \int_0^\infty t^{(n-1)/2}\exp(-t)\,\mathrm{d}t \\
&= \Gamma\left(\frac{n+1}{2}\right).
\end{aligned}$$

□

The next lemma establishes the behaviour of integrals of monomials over the surface of a sphere. We will use it in the main proof when we write a Taylor expansion of f around x, and evaluate the integral of each monomial separately.

Lemma 3. *When m_i is an even natural number for i between 1 and n,*

$$\int_{S(0,1)} \prod_{i=1}^{n} y_i^{m_i}\,\mathrm{d}\sigma(y) = \frac{2\prod_{i=1}^{n}\Gamma((m_i+1)/2)}{\Gamma\left(\sum_{i=1}^{n}((m_i+1)/2)\right)}. \tag{1}$$

When any m_i is odd, the left-hand side of equation 1 vanishes.

Proof. Adapted from [2]. First: when any m_i is odd, the left-hand side of equation 1 vanishes by symmetry, so we need only consider the case when each m_i is even. Consider the integral

$$I = \int_{\mathbb{R}^n} \prod_i y_i^{m_i} \exp\left(-|y|^2\right)\mathrm{d}y$$

We will find our result by evaluating I in rectangular and polar coordinates, and comparing the results. First, in rectangular coordinates,

$$\begin{aligned}
I &= \int_{\mathbb{R}^n} \prod_i \left(y_i^{m_i}\exp\left(-y_i^2\right)\right)\mathrm{d}y \\
&= \prod_i \int_{-\infty}^{\infty} y_i^{m_i}\exp\left(-y_i^2\right)\mathrm{d}y_i
\end{aligned}$$

Since m_i is even, we can restrict our domain to $[0,\infty)$:

$$\begin{aligned}
&= \prod_i 2\int_0^\infty y_i^{m_i}\exp\left(-y_i^2\right)\mathrm{d}y_i \\
&= \prod_i \Gamma\left(\frac{m_i+1}{2}\right), \qquad (2)
\end{aligned}$$

using lemma 2.

Next, we evaluate I in polar coordinates, by integrating over rays thru the unit sphere $S(0,1)$. Given an infinitesimal patch of area $\mathrm{d}\sigma$ on the unit sphere, once that patch is projected out to a distance r from the origin, its $(n-1)$-volume is multiplied by r^{n-1}, and it has thickness $\mathrm{d}r$, meaning that the differential unit of volume will be $r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma$. So,

$$\begin{aligned}
I &= \int_{S(0,1)} \int_0^\infty \prod_i (ry_i)^{m_i} \exp\left(-r^2\right) r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma(y) \\
&= \left(\int_0^\infty r^{(\sum_i m_i)+n-1} \exp\left(-r^2\right)\mathrm{d}r\right)\left(\int_{S(0,1)} \prod_i y_i^{m_i}\,\mathrm{d}\sigma(y)\right) \\
&= \frac{1}{2}\Gamma\left(\sum_i \frac{m_i+1}{2}\right) \int_{S(0,1)} \prod_i y_i^{m_i}\,\mathrm{d}\sigma(y), \qquad (3)
\end{aligned}$$

again using lemma 2.

Comparing equation 2 and equation 3, we get equation 1. □

We are now ready to prove our main theorem.

Proof of theorem 1. First, note that based on the scaling of the surface volume of spheres, $\mathrm{Vol}_{n-1}(S(x,r)) = r^{n-1}\mathrm{Vol}_{n-1}(S(0,1))$, which we can evaluate as the integral of the constant monomial:

$$\begin{aligned}
\mathrm{Vol}_{n-1}(S(0,1)) &= \int_{S(0,1)} \prod_i y_i^0\,\mathrm{d}\sigma(y) \\
&= \frac{2\left(\Gamma(1/2)\right)^n}{\Gamma(n/2)} \\
&= \frac{2\pi^{n/2}}{\Gamma(n/2)}. \qquad (4)
\end{aligned}$$

Next, note that we can use Taylor's theorem to write

$$\begin{aligned}
f(x') = f(x) &+ \sum_i \frac{\partial f(x)}{\partial x_i}(x_i' - x_i) + \sum_{i<j} \frac{\partial^2 f(x)}{\partial x_i \partial x_j}(x_i' - x_i)(x_j' - x_j) \\
&+ \sum_i \frac{1}{2}\frac{\partial^2 f(x)}{\partial x_i^2}(x_i' - x_i)^2 + h(x')\sum_i (x_i' - x_i)^2,
\end{aligned}$$

where $h(x') \to 0$ as $x' \to x$.

We can integrate this Taylor expansion over $S(x,r)$ using lemma 3:

$$\begin{aligned}
&\int_{S(x,r)} f(x')\,\mathrm{d}\sigma(x') \\
&= \int_{S(0,r)} f(x+y')\,\mathrm{d}\sigma(y')
\end{aligned}$$

Let $y = y'/r$. Then $\mathrm{d}\sigma(y') = r^{n-1}\,\mathrm{d}\sigma(y)$.

$$
\begin{aligned}
&= \int_{S(0,1)} f(x+ry) r^{n-1}\,\mathrm{d}\sigma(y) \\
&= r^{n-1}\int_{S(0,1)} f(x)\,\mathrm{d}\sigma(y) + \sum_i \frac{\partial f(x)}{\partial x_i} r^n \int_{S(0,1)} y_i\,\mathrm{d}\sigma(y) \\
&\quad + \sum_{i<j} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} r^{n+1} \int_{S(0,1)} y_i y_j\,\mathrm{d}\sigma(y) + \sum_i \frac{1}{2}\frac{\partial^2 f(x)}{\partial x_i^2} r^{n+1} \int_{S(0,1)} y_i^2\,\mathrm{d}\sigma(y) \\
&\quad + r^{n+1}\int_{S(0,1)} h(x+ry) \sum_i y_i^2\,\mathrm{d}\sigma(y) \\
&= f(x) r^{n-1}\mathrm{Vol}_{n-1}(S(0,1)) + 0 + 0 + \sum_i \frac{1}{2}\frac{\partial^2 f(x)}{\partial x_i^2} r^{n+1} \frac{2\Gamma(3/2)\Gamma(1/2)^{n-1}}{\Gamma(1+n/2)} \\
&\quad + r^{n+1}\int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y) \\
&= f(x)\mathrm{Vol}_{n-1}(S(x,r)) + r^{n+1}\frac{(1/2)\Gamma(1/2)^n}{(n/2)\Gamma(n/2)}\Delta f(x) \\
&\quad + r^{n+1}\int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y) \\
&= f(x)\mathrm{Vol}_{n-1}(S(x,r)) + r^{n+1}\frac{\pi^{n/2}}{n\Gamma(n/2)}\Delta f(x) + r^{n+1}\int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y).
\end{aligned}
\tag{5}
$$

Therefore, we can now evaluate the main expression, using equation 5 to evaluate the integral and equation 4 to evaluate $\mathrm{Vol}_{n-1}(S(x,r))$:

$$
\begin{aligned}
&\lim_{r\to 0}\frac{2n}{r^2}\left(\frac{\int_{S(x,r)} f(x')\,\mathrm{d}\sigma(x')}{\mathrm{Vol}_{n-1}(S(x,r))} - f(x)\right) \\
&= \lim_{r\to 0}\frac{2n}{r^2}\left(f(x) + r^{n+1}\frac{\pi^{n/2}}{n\Gamma(n/2)\mathrm{Vol}_{n-1}(S(x,r))}\Delta f(x)\right. \\
&\quad \left. + \frac{r^{n+1}}{\mathrm{Vol}_{n-1}(S(x,r))}\int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y) - f(x)\right) \\
&= \lim_{r\to 0}\frac{2n}{r^2}\left(\frac{r^{n+1}\pi^{n/2}}{n\Gamma(n/2) r^{n-1} 2\pi^{n/2}/\Gamma(n/2)}\Delta f(x)\right. \\
&\quad \left. + \frac{r^{n+1}}{r^{n-1}2\pi^{n/2}/\Gamma(n/2)}\int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y)\right) \\
&= \lim_{r\to 0}\frac{2n}{r^2}\left(\frac{r^2}{2n}\Delta f(x) + \frac{r^2\Gamma(n/2)}{2\pi^{n/2}}\int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta f(x) + \lim_{r\to 0} \frac{n\Gamma(n/2)}{\pi^{n/2}} \int_{S(0,1)} h(x+ry)\,\mathrm{d}\sigma(y) \\
&= \Delta f(x),
\end{aligned}
$$

since as $r \to 0$, $h(x+ry) \to 0$.

$\square$

References

[1] Fan RK Chung. *Spectral graph theory*. 92. American Mathematical Soc., 1997.

[2] Gerald B Folland. "How to integrate a polynomial over a sphere". In: *The American Mathematical Monthly* 108.5 (2001), pp. 446–448.